C4 Exercise 6E (standard patterns in integration)

This exercise uses two key general patterns

- To integrate expressions of the form $\int k \frac{f'(x)}{f(x)}\,dx$ try using $\ln|f(x)|$ and differentiate it to check and adjust any constant.
- To integrate an expression of the form $\int k f'(x)[f(x)]^n\,dx$ try using $[f(x)]^{n+1}$ and differentiate to check and adjust any constant.

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Example 10 a

$$\int \frac{\operatorname{cosec}^2 x}{(2+\cot x)^3}\,dx \equiv \int \operatorname{cosec}^2 x\,(2+\cot x)^{-3}\,dx$$

now since $\frac{d}{dx}(2+\cot x) = \operatorname{cosec}^2 x$

the integral is in the form $\int f'(x)\,[f(x)]^n\,dx$

so I'll try $[f(x)]^{n+1}$, namely $(2+\cot x)^{-2}$

now putting $u = 2+\cot x \Rightarrow \frac{du}{dx} = -\operatorname{cosec}^2 x$, using $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

we have $\frac{d}{dx}(2+\cot x)^{-2} = -2u^{-3}(-\operatorname{cosec}^2 x)$

$$= 2\operatorname{cosec}^2 x\,(2+\cot x)^{-3}$$

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Example 10a ctd

so if we start with $\frac{1}{2}(2+\cot x)^{-2}$ we will get $\operatorname{cosec}^2 x(2+\cot x)^{-3}$

Hence $\displaystyle\int \frac{\operatorname{cosec}^2 x}{2+\cot x}\,dx = \frac{1}{2}(2+\cot x)^{-2} + C$

Example 10b

$$\int 5\tan x \sec^4 x \, dx$$

now $\frac{d}{dx}\sec x = \sec x \tan x$ so we can rearrange the integral to say $\int 5(\sec x \tan x)\sec^3 x\,dx$

which is now of the form $\int k\, f'(x)[f(x)]^n\,dx$

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Example 10b (ctd) and we should try using $[f(x)]^{n+1}$.

$$\frac{d}{dx}\sec^4 x = 4(\sec x \tan x)\sec^3 x \; dx$$

but since we want $5(\sec x \tan x)\sec^3 x$ we shall need

to start with $\frac{5}{4}\sec^4 x$ hence:

$$\int 5\tan x \sec^4 x \, dx = \frac{5}{4}\sec^4 x + C$$

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1a $\displaystyle\int \frac{x}{x^2+4}\,dx \equiv \int x(x^2+4)^{-1}\,dx$

Note that if $f(x) \equiv x^2+4$ then $f'(x) = x$

Try $\dfrac{d}{dx} \ln|x^2+4| = 2x \times \dfrac{1}{x^2+4}$

$\therefore \displaystyle\int \frac{x}{x^2+4}\,dx = \frac{1}{2}\ln|x^2+4| + C$

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$$\int \frac{e^{2x}}{e^{2x}+1}\,dx \equiv \int e^{2x}\left(e^{2x}+1\right)^{-1} dx$$

So it's of the form $\int \frac{f'(x)}{f(x)}\,dx$ and we try $\ln|f(x)|$

$$\frac{d}{dx}\left(\ln|e^{2x}+1|\right) = 2e^{2x}\left(e^{2x}+1\right)^{-1}$$

so $$\int \frac{e^{2x}}{e^{2x}+1}\,dx = \frac{1}{2}\ln|e^{2x}+1| + C$$

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1c $\int \frac{x}{(x^2+4)^3}\,dx \equiv \int x(x^2+4)^{-3}\,dx$ and is of form $\int k f'(x)[f(x)]^n\,dx$

so try $[f(x)]^{n+1}$:

$$\frac{d}{dx}(x^2+4)^{-2} = -2 \times 2x(x^2+4)^{-3} = -4x(x^2+4)^{-3}$$

so $\int \frac{x}{(x^2+4)^3}\,dx = -\frac{1}{4}(x^2+4)^{-2} + c$

$= -\frac{1}{4(x^2+4)^2} + c$ if you prefer

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1d $\displaystyle\int \frac{e^{2x}}{(e^{2x}+1)^3}\,dx \equiv \int e^{2x}(e^{2x}+1)^{-3}\,dx$ and of form $\int k\,f'(x)[f(x)]^n\,dx$

Try $(e^{2x}+1)^{-2}$.

$$\frac{d}{dx}(e^{2x}+1)^{-2} = -2 \times 2e^{2x} \times (e^{2x}+1)^{-3}$$

Hence $\displaystyle\int \frac{e^{2x}}{(e^{2x}+1)^3}\,dx = -\frac{1}{4}(e^{2x}+1)^{-2} + C$

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1e $\displaystyle\int \frac{\cos 2x}{3+\sin 2x}\,dx \equiv \int \cos 2x\,(3+\sin 2x)^{-1}\,dx$

and is of the form $\int k f'(x)[f(x)]^{-1}\,dx$ so try $\ln|f(x)|$

Try $\dfrac{d}{dx}\ln|3+\sin 2x| = 2\cos 2x\,(3+\sin 2x)^{-1}$

So $\displaystyle\int \frac{\cos 2x}{3+\sin 2x}\,dx = \frac{1}{2}\ln|3+\sin 2x| + C$

C4Ex6E

1f $\displaystyle\int \frac{\sin 2x}{(3+\cos 2x)^3}\,dx \equiv \int \sin 2x\,(3+\cos 2x)^{-3}\,dx$

and is of form $\int k\, f'(x)[f(x)]^n\,dx$ so try $\frac{d}{dx}[f(x)]^{n+1}$

$\frac{d}{dx}(3+\cos 2x)^{-2} = -2(-2\sin 2x)(3+\cos 2x)^{-3}$

So $\displaystyle\int \frac{\sin 2x}{(3+\cos 2x)^3}\,dx = \frac{1}{4}(3+\cos 2x)^{-2} + C$

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1g $\int xe^{x^2}\,dx$ is of the form $\int k\,f'(x)\,[f(x)]^n\,dx$

since $\frac{d}{dx} e^{x^2} = \frac{de^u}{du} \cdot \frac{du}{dx} = e^u \times 2x = 2xe^{x^2}$

so $\int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + C$

C4 Ex 6E

1h $\int \cos 2x(1 + \sin 2x)^4 \, dx$

note $\frac{d}{dx}(1+\sin 2x)^5 = 5 \times (2\cos 2x)(1+\sin 2x)^4$

so $\int \cos 2x \,(1 + \sin 2x)^4 \, dx = \frac{1}{10}(1+\sin 2x)^5 + C$

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1i $\int \sec^2 x \tan^2 x \, dx$

$$\frac{d}{dx}\tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{\cos x \cos x + \sin x \sin x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

So $\int \sec^2 x \tan^2 x \, dx$ is of form $\int k f'(x)\left[f(x)\right]^n dx$

with $f(x) \equiv \tan x$ so try $\frac{d}{dx}\tan^3 x = 3(\sec^2 x)(\tan^2 x)$

so $\int \sec^2 x \tan^2 x = \frac{1}{3}\tan^3 x + C$

C4 Ex 6E

1j $\int \sec^2 x\,(1+\tan^2 x)\,dx \equiv \int \sec^2 x\,dx + \int \sec^2 x \tan^2 x\,dx$

now $\frac{d}{dx}\tan x = \sec^2 x$ (see previous page)

and $\frac{d}{dx}\tan^3 x = 3\sec^2 x \tan^2 x$ (also in question 1i)

So $\int \sec^2 x\,(1+\tan^2 x)\,dx = \tan x + \frac{1}{3}\tan^3 x + C$

C4 Ex 6E

2a

$$\int (x+1)(x^2+2x+3)^4\,dx$$

notice $\frac{d}{dx}\, x^2+2x+3 = 2x+2$ which is $2(x+1)$

so $\int (x+1)(x^2+2x+3)\,dx$ is of form $\int k\, f'(x)\left[f(x)\right]^n dx$

with $f(x) \equiv x^2+2x+3$ and $n=4$

so try $\frac{d}{dx}(x^2+2x+3)^5 = 5(2x+2)(x^2+2x+3)^4$

$$= 10(x+1)(x^2+2x+3)^4$$

so $\int (x+1)(x^2+2x+3)\,dx = \frac{1}{10}(x^2+2x+3)^4 + C$

C4 Ex 6E

2b $\int \operatorname{cosec}^2 2x \cot 2x \, dx$

Recall $\frac{d}{dx} \cot 2x \equiv \frac{d}{du}\left(\frac{\cos u}{\sin u}\right)\frac{du}{dx}$, etc...

$= -2\operatorname{cosec}^2 2x$

so $\int \operatorname{cosec}^2 2x \cot 2x \, dx$ is of form $\int k\, f'(x)\,[f(x)]^n \, dx$

so try $[f(x)]^{n+1}$ ie $\frac{d}{dx} \cot^2 2x = 2(-2\operatorname{cosec}^2 2x)\cot 2x$

$= -4\, f'(x)\,[f(x)]^n$

So $\int \operatorname{cosec}^2 2x \cot 2x \, dx = -\frac{1}{4}\cot^2 2x + C$

C4 Ex 6E

2b $\displaystyle\int \operatorname{cosec}^2 2x \cot 2x \, dx$ alternative method

re-write as $\displaystyle\int \underbrace{\operatorname{cosec} 2x \cot 2x}_{= k f'(x)} \underbrace{\operatorname{cosec} 2x}\, dx$

$= k f'(x)$ when $f(x) = \operatorname{cosec} 2x$

so is of form $\int k f'(x) [f(x)]^n \, dx$ with $f(x) = \operatorname{cosec} 2x$, $n=1$

so try $\dfrac{d}{dx}[f(x)]^{n+1} = \dfrac{d}{dx} \operatorname{cosec}^2 2x = 2(-2\operatorname{cosec} x \cot x)\operatorname{cosec} 2x$

$= -4 \, f'(x) \, [f(x)]^n$

so $\displaystyle\int \operatorname{cosec}^2 2x \cot 2x \, dx = -\frac{1}{4}\operatorname{cosec}^2 2x + C$

How can $I = -\frac{1}{4}\cot^2 2x + C$

and $I = -\frac{1}{4}\operatorname{cosec}^2 2x + C$?

Well recall that $\sin^2 u + \cos^2 u \equiv 1$

So $\frac{1}{\cancel{\sin^2 u}}\cancel{\frac{\sin^2 u}{}} \; \frac{1 \quad \cancel{\sin^2 u}}{\cancel{\sin^2 u}} + \cot^2 u \equiv \operatorname{cosec}^2 u$

Hence $-\frac{1}{4}\cot^2 u + {}^-1/_4 \equiv -\frac{1}{4}\operatorname{cosec}^2 u$

So $-\frac{1}{4}\cot^2 2x + C_1 \equiv -\frac{1}{4}\operatorname{cosec}^2 2x + C_2$

provided $C_1 - C_2 = {}^-1/_4$

So these solutions differ only by their constant.

C4 Ex 6E

2c $I = \int \sin^5 3x \cos 3x \, dx$ is of form $\int k f'(x)[f(x)]^n \, dx$

so try $\frac{d}{dx}[f(x)]^{n+1}$ where $f(x) = \sin 3x$ and $n = 5$

$$\frac{d}{dx} \sin^6 3x = 6(3\cos 3x)\sin^5 3x$$

$$= 18 \sin^5 3x \cos 3x$$

so $I = \frac{1}{18} \sin^6 3x + C$

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2d $I = \int \cos x \, e^{\sin x} \, dx$

consider $\frac{d}{dx} e^{\sin x} = \frac{d}{du} e^{u} \, \frac{du}{dx}$ where $u = \sin x$
(chain rule)

$= e^{u} \cdot \cos x = \cos x \, e^{\sin x}$

so $I = e^{\sin x} + C$

C4 Ex 6E

2e $\quad I = \int \frac{e^{2x}}{e^{2x}+3}\,dx$

try $\frac{d}{dx} \ln|e^{2x}+3| = 2e^{2x} \ln|e^{2x}+3|$

so $\quad I = \frac{1}{2} \ln|e^{2x}+3| + C$

C4 Ex6E

2f $I = \int x(x^2+1)^{3/2}\,dx$ is of form $\int k\,f'(x)\,[f(x)]^n\,dx$

where $f(x) = x^2+1$ and $n = 3/2$

try $\frac{d}{dx}[f(x)]^{n+1}$ ie $\frac{d}{dx}(x^2+1)^{5/2} = \frac{5}{2}(2x)(x^2+1)^{3/2}$

so since this is 5 times what I wish to integrate

$I = \frac{1}{5}(x^2+1)^{5/2} + C$

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2g $I = \int (2x+1)\sqrt{x^2+x+5}\,dx = \int (2x+1)(x^2+x+5)^{1/2}\,dx$

is of form $\int k\, f'(x)\,[f(x)]^n$ with $f(x) = x^2+x+5$ and $n = 1/2$

so try $\frac{d}{dx}[f(x)]^{n+1} = \frac{d}{dx}(x^2+x+5)^{3/2} = \frac{3}{2}(2x+1)(x^2+x+5)^{1/2}$

hence $I = \frac{2}{3}(x^2+x+5)^{3/2}$

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2h

$$I = \int \frac{2x+1}{\sqrt{x^2+x+5}}\,dx = \int (2x+1)(x^2+x+5)^{-1/2}\,dx$$

now $$\frac{d}{dx}(x^2+x+5)^{1/2} = \frac{1}{2}(2x+1)(x^2+x+5)^{-1/2}$$

so $$I = 2(x^2+x+5)^{1/2} + C$$

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2i

$$I = \int \frac{\sin x \cos x}{\sqrt{\cos 2x + 3}}\,dx = \int (\sin x \cos x)(\cos 2x + 3)^{-1/2}\,dx$$

At first glance this does not appear to fit the pattern $\int k\, f'(x)\,[f(x)]^n\,dx$, but in fact consider

$$\frac{d}{dx}(\cos 2x + 3)^{1/2} = \tfrac{1}{\cancel{2}}[-\cancel{2}\sin 2x][\cos 2x + 3]^{-1/2}$$

now $\sin 2x = 2\sin x \cos x$

so $I = \int \tfrac{1}{2}\sin 2x\,[\cos 2x + 3]^{-1/2}$

so $I = -\tfrac{1}{2}(\cos 2x + 3)^{1/2} + C$

C4 Ex6E

2j

$$I = \int \frac{\sin x \cos x}{\cos 2x + 3}\,dx = \int \frac{\frac{1}{2}\sin 2x}{\cos 2x + 3}\,dx$$

try $\frac{d}{dx} \ln|\cos 2x + 3| = (-2\sin 2x)\left(\frac{1}{\cos 2x + 3}\right)$

so $I = -\frac{1}{4} \ln|\cos 2x + 3| + C$